

Weyl structures with positive Ricci tensor

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Abstract

We prove the vanishing of the first Betti number on compact manifolds admitting a Weyl structure whose Ricci tensor satisfies certain positivity conditions, thus obtaining a Bochner-type vanishing theorem in Weyl geometry. We also study compact Hermitian-Weyl manifolds with non-negative symmetric part of the Ricci tensor of the canonical Weyl connection and show that every such manifold has first Betti number $b_1 = 1$ and Hodge numbers $h^{p,0} = 0$ for $p > 0$, $h^{0,1} = 1$, $h^{0,q} = 0$ for $q > 1$.

Running title: Weyl structures with positive Ricci tensor

Keywords. Weyl structure, Hermitian-Weyl structure, generalized Hopf manifold, rational surface

MS classification: 53C15; 53C55

*The authors are supported by Contract MM 809/1998 with the Ministry of Science and Education of Bulgaria and by Contract 238/1998 with the University of Sofia "St. Kl. Ohridski".

1 Introduction

A Weyl structure on a conformal manifold (M, c) is a torsion-free linear connection ∇^W preserving the conformal structure c . This implies that for any Riemannian metric $g \in c$ there exists a 1-form θ such that $\nabla^W g = \theta \otimes g$. Conversely, given a 1-form θ and $g \in c$, there exists a unique Weyl structure ∇^W such that $\nabla^W g = \theta \otimes g$. An example of a Weyl structure is the Levi-Civita connection of a metric in c . This is the case when θ is exact and such Weyl structures are called exact. Similarly, ∇^W is called closed if θ is closed, i.e., if ∇^W locally coincides with the Levi-Civita connection of a metric in c . The well known theorem of Gauduchon [8] says that if M is compact and at least 3-dimensional, then for every Weyl structure there exists a unique (up to homothety) metric $g \in c$, such that the corresponding 1-form θ is co-closed with respect to g .

Weyl structures arise naturally for example in almost Hermitian and contact geometry (the connection is determined by the Lee form and the contact form respectively). More generally, when the geometry has a preferred 1-form, then the Weyl structure is determined by the given metric and this 1-form.

Recently there has been considerable interest in Weyl geometry, mainly in Einstein-Weyl manifolds. A Weyl structure is called Einstein-Weyl if the symmetric trace-free part of its Ricci tensor vanishes. The Einstein-Weyl geometry initially attracted particular interest in dimension 3 [20, 21], but subsequently Einstein-Weyl manifolds have been much studied in all dimensions [12, 18, 19] (for a nice overview on Einstein-Weyl manifolds see [6]).

In the present paper we study Weyl structures on compact manifolds, such that the symmetric part of the Ricci tensor satisfies certain positivity conditions.

In [11] Gauduchon showed that a 4-dimensional compact conformal manifold, which admits a closed self-dual Weyl structure with 2-positive normalized Ricci tensor, is conformally equivalent to S^4 or CP^2 with their standard conformal structures. Notice that the 2-positivity condition of Gauduchon implies the positivity of the Ricci tensor of the Weyl connection but the converse is not true. In [19] Pedersen and Swann proved that on a non-exact compact Einstein-Weyl manifold with vanishing (resp. non-negative but non-zero) symmetric part of the Ricci tensor the first Betti number is $b_1 = 1$ (resp. $b_1 = 0$). Recently the authors showed [2] that a compact Hermitian surface with non-negative and positive at some point symmetric part of the Ricci tensor of the canonical Weyl connection has $b_1 = 0$.

In the next theorem we prove that $b_1 = 0$ for any compact conformal manifold, which admits a Weyl structure such that the symmetric part of the Ricci tensor satisfies certain positivity conditions. Thus we obtain a vanishing theorem of Bochner type in Weyl geometry.

Theorem 1.1 *Let (M, c) be a compact n -dimensional ($n > 2$) conformal manifold, ∇^W a non-exact Weyl connection on (M, c) and $g \in c$ the Gauduchon metric of ∇^W . Let θ be the 1-form determined by ∇^W and g and $\theta^\#$ the vector field dual to θ with respect to g . Suppose the symmetric part Ric^W of the Ricci tensor of ∇^W satisfies*

$$(1.1) \quad Ric^W(X, X) \geq \frac{(n-2)(n-4)}{8}(|\theta|^2|X|^2 - \theta(X)^2)$$

for all tangent vectors X . Then $b_1 \leq 1$. Further,

a) If at some point Inequality (1.1) is strict for $X = \theta^\#$ (i.e. $Ric^W(\theta^\#, \theta^\#) > 0$), then $b_1(M) = 0$.

b) If $b_1 = 1$, then θ is parallel with respect to the Levi-Civita connection of g and the universal cover of (M, g) is isometric to $\mathbf{R} \times N$, where the metric on N has positive Ricci curvature. In particular, for $n = 3$ and $n = 4$ the manifold N is diffeomorphic to S^{n-1} .

Note that if $n = 4$ Condition (1.1) is reduced to $\text{Ric}^W(X, X) \geq 0$ and if $n = 3$ its right-hand side is even non-positive. In Corollary 2.1 we also show that any oriented 3-dimensional manifold, which satisfies the assumptions of Theorem 1.1 and has $b_1 = 1$, is diffeomorphic to $S^1 \times S^2$.

Recall that the canonical Weyl structure on a Hermitian manifold is determined by the metric and the Lee form. The manifold is called Hermitian-Weyl if the canonical Weyl structure preserves the complex structure (see [17]). This condition is always satisfied in the 4-dimensional case, but for higher dimensions it forces the manifold to be locally conformally Kähler. Every locally conformally Kähler manifold has closed Lee form and a particular subclass of such manifolds is formed by the generalized Hopf manifolds, the Hermitian manifolds whose Lee form is parallel with respect to the Levi-Civita connection. Compact Hermitian Einstein-Weyl manifolds are classified in [13] in dimension 4 and the higher dimensional case is studied in detail in [17].

We apply Theorem 1.1 to Hermitian Weyl four-manifolds to get

Corollary 1.2 *Let (M, J) be a compact complex surface. Suppose there exists a conformal class of Hermitian metrics such that the corresponding canonical Weyl structure is non-exact and has Ricci tensor with everywhere non-negative symmetric part. Then $b_1 \leq 1$ and*

- (i) *If $b_1 = 0$, then (M, J) is a rational surface with $c_1^2 \geq 0$;*
- (ii) *If $b_1 = 1$, then (M, J) is a Hopf surface of class 1.*

Note that if the symmetric part of the Ricci tensor of the canonical Weyl structure is non-negative everywhere and strictly positive at some point, then the surface is rational with $c_1^2 > 0$ [2].

In dimensions greater than four the non-exact Hermitian-Weyl manifolds with non-negative symmetric part of the Ricci tensor are generalized Hopf manifolds with respect to the Gauduchon metric by the results in [25]. It is well known that the first Betti number of a generalized Hopf manifold is odd and therefore these manifolds do not admit any Kähler structure. In general, the first Betti number and the Hodge numbers of a generalized Hopf manifold are not known (for surveys on generalized Hopf manifolds see [26, 7]). We prove

Theorem 1.3 *Let (M, g, J) be a $2m$ -dimensional compact generalized Hopf manifold and the canonical Hermitian Weyl structure has Ricci tensor with non-negative symmetric part. Then $b_1 = 1$ and the Hodge numbers $h^{p,q}$ satisfy*

$$h^{p,0} = 0, \quad p = 1, 2, \dots, m, \quad h^{0,q} = 0, \quad q = 2, 3, \dots, m, \quad h^{0,1} = 1.$$

2 Ricci-positive Weyl structures in dimension n

In the following we denote by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ the pointwise inner products and norms and by (\cdot, \cdot) and $\|\cdot\|$ – the global ones respectively. For the curvature and the Ricci tensor of a linear connection ∇ we adopt the following convention: $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}$, $\rho(X, Y) = \text{trace}\{Z \longrightarrow R(Z, X)Y\}$.

Let ∇^W be a Weyl structure on an n -dimensional conformal manifold (M, c) . Let $g \in c$ and $\nabla^W g = \theta \otimes g$. The connection ∇^W is given explicitly by

$$(2.2) \quad \nabla_X^W Y = \nabla_X Y - \frac{1}{2}\theta(X)Y - \frac{1}{2}\theta(Y)X + \frac{1}{2}g(X, Y)\theta^\#,$$

where ∇ is the Levi-Civita connection of g . We shall denote by Ric^W the symmetric part of the Ricci tensor of ∇^W and by k the conformal scalar curvature, i.e. the trace of Ric^W with respect to g . It is easy to obtain from (2.2) that the whole Ricci tensor of ∇^W is $Ric^W + \frac{n}{4}d\theta$,

$$(2.3) \quad Ric^W(X, X) = \rho(X, X) + \frac{n-2}{2}(\nabla_X \theta)(X) - \frac{n-2}{4}(|\theta|^2|X|^2 - \theta(X)^2) - \frac{1}{2}|X|^2 d^* \theta,$$

$$(2.4) \quad k = s - (n-1)d^* \theta - \frac{(n-1)(n-2)}{4}|\theta|^2,$$

where ρ and s are the Ricci tensor and the scalar curvature of the Levi-Civita connection of g and the norms and the codifferential operator are taken with respect to g .

Proof of Theorem 1.1: Let $\tilde{\nabla}$ be the connection defined by

$$(2.5) \quad \tilde{\nabla}_X Y = \nabla_X Y - \frac{n-2}{4}(\theta(X)Y - g(X, Y)\theta^\#).$$

Let φ be a 1-form and ξ the dual vector field. It follows from (2.5) that

$$(2.6) \quad |\tilde{\nabla}\xi|^2 = |\nabla\xi|^2 - \frac{n-2}{2} \langle \nabla_{\theta^\#} \xi, \xi \rangle + \frac{n-2}{2}\theta(\nabla_\xi \xi) + \frac{(n-2)^2}{8}(|\theta|^2|\xi|^2 - \theta(\xi)^2).$$

Since

$$(\nabla_{\theta^\#} \xi, \xi) = \frac{1}{2}(\theta, d(|\xi|^2)) = \frac{1}{2} \int_M |\xi|^2 d^* \theta dV = 0,$$

by (2.3), (2.6) and the well-known Weitzenböck formula

$$\|d\varphi\|^2 + \|d^* \varphi\|^2 = \|\nabla \xi\|^2 + \int_M \rho(\xi, \xi) dV$$

(see e.g. [5], Chapter 1) we obtain

$$(2.7) \quad \|d\varphi\|^2 + \|d^* \varphi\|^2 = \|\tilde{\nabla} \xi\|^2 + \int_M (Ric^W(\xi, \xi) - \frac{(n-2)(n-4)}{8}(|\theta|^2|\xi|^2 - \theta(\xi)^2)) dV.$$

Now, let $\varphi \neq 0$ be a harmonic 1-form. Then (2.7) and (1.1) show that $\tilde{\nabla} \xi = 0$. Hence, $\nabla \varphi = \frac{n-2}{4}\theta \wedge \varphi$ is skew-symmetric. But the skew-symmetric part of $\nabla \varphi$ is $d\varphi = 0$. Therefore $\theta = f\varphi$ for some smooth function f and $\nabla \varphi = 0$. Thus, the universal cover of (M, g) is isometric to $\mathbf{R} \times N$ (see e.g. chapter 4 in [15]). We can lift θ , φ , ξ and f to $\mathbf{R} \times N$ and we can assume that $|\varphi| = 1$, i.e. ξ is the unit vector field tangent to \mathbf{R} . Since θ and φ are both co-closed, from $\theta = f\varphi$ we get $df(\xi) = 0$. By (2.3) and (1.1) we have

$$(2.8) \quad \rho(X, X) + \frac{n-2}{2}\varphi(X)df(X) \geq \frac{(n-2)^2}{8}f^2(|\varphi|^2|X|^2 - \varphi(X)^2).$$

Any vector tangent to $\mathbf{R} \times N$ has the form $X = \lambda\xi + X^\perp$, where X^\perp is orthogonal to ξ , and since the metric on $\mathbf{R} \times N$ is the product metric, $\rho(X, X) = \rho(X^\perp, X^\perp)$. Thus, using (2.8) we obtain

$$(2.9) \quad \rho(X^\perp, X^\perp) + \frac{n-2}{2}\lambda df(X^\perp) \geq \frac{(n-2)^2}{8}f^2|X^\perp|^2$$

for any $\lambda \in \mathbf{R}$. Hence, for any vector X^\perp tangent to N we have $df(X^\perp) = 0$. This together with $df(\xi) = 0$ means that f is constant and therefore θ is parallel. Since the Weyl structure on M is not exact, we have $f \neq 0$ (otherwise $\theta = 0$). Thus, any harmonic form on (M, g) is a constant multiple of θ and so if $b_1 \neq 0$, then $b_1 = 1$. It follows from (2.9) that the Ricci curvature of N is positive.

When $n = 4$, N is a compact simply connected 3-dimensional manifold admitting a metric of positive Ricci curvature and by a theorem of Hamilton (cf. [14] or Theorem 5.30 in [5]) N is diffeomorphic to S^3 . When $n = 3$, N is a compact simply connected 2-dimensional manifold admitting a metric of positive Ricci curvature (i.e. of positive Gauss curvature) and therefore is diffeomorphic to S^2 . Thus b) is proved.

If Inequality (1.1) is strict for $X = \theta^\#$ at some point, then it follows from (2.7) that θ is not a harmonic form. This means that there are no harmonic 1-forms, which proves a). **Q.E.D.**

Corollary 2.1 *Any oriented 3-dimensional manifold M , which satisfies the assumptions of Theorem 1.1 and has $b_1 \neq 0$, is diffeomorphic to $S^1 \times S^2$.*

Proof: By Theorem 1.1 the universal cover of M is isometric to $\mathbf{R} \times N$, where N is diffeomorphic to S^2 . Any metric on S^2 is conformal to the standard one and therefore orientation-preserving isometries of N are contained in $SL(2, \mathbf{C})$. Now, by arguments similar to those in the proof of Theorem 3.2 in [1] we obtain that the fundamental group of M is isomorphic to \mathbf{Z} and M is the total space of a locally trivial fibre bundle over S^1 with fibre S^2 and structure group contained in $SL(2, \mathbf{C})$. Hence, M is diffeomorphic to $S^1 \times S^2$. **Q.E.D.**

The following result is proved in [19] for Einstein-Weyl manifolds.

Corollary 2.2 *If (M, c) is a compact 4-dimensional conformal manifold with a Weyl structure whose Ricci tensor is non-negative everywhere and positive at some point symmetric part, then the 1-form θ given by any choice of a compatible metric must vanish somewhere.*

Proof: It follows from Theorem 1.1 that $b_1 = 0$ and therefore the Euler characteristic of M is positive. Hence, every 1-form on M vanishes somewhere. **Q.E.D.**

3 Ricci-positive Hermitian-Weyl structures

Let (M, g, J) be a $2m$ -dimensional ($m > 1$) almost Hermitian manifold with almost complex structure J and compatible metric g . Let Ω be the Kähler form of (M, g, J) , defined by $\Omega(X, Y) = g(X, JY)$. Denote by θ the Lee form of (M, g, J) , $\theta = -\frac{1}{m-1}Jd^*\Omega$ (for a 1-form α , we define $J\alpha = -\alpha \circ J$).

The canonical Weyl structure ∇^W corresponding to the almost Hermitian structure (g, J) on M is determined by the metric g and its Lee form θ . The canonical Weyl structure does not depend on the choice of the metric in the conformal class c of g , i.e. the Weyl structure determined by a metric $\tilde{g} \in c$ and its Lee form $\tilde{\theta}$ coincides with ∇^W .

The canonical Weyl structure is called Hermitian-Weyl [17] if it preserves J . Since ∇^W is torsion-free, this shows that J must be integrable. According to the results in [23], the canonical Weyl structure on a Hermitian manifold (M, g, J) is Hermitian-Weyl iff

$$(3.10) \quad d\Omega = \theta \wedge \Omega.$$

This condition is always satisfied in the 4-dimensional case. For higher dimensions it means that the Hermitian manifold (M, g, J) is locally conformally Kähler. In particular, $d\theta = 0$. Notice

that since the antisymmetric part of the Ricci tensor of ∇^W is $\frac{m}{2}d\theta$, on locally conformally Kähler manifolds the Ricci tensor of the canonical Weyl connection is symmetric.

Proof of Corollary 1.2. Theorem 1.1 implies $b_1 \leq 1$. Since the Weyl structure is not exact, (2.4) shows that the Gauduchon metric has non-negative but not identically zero scalar curvature. Thus, by Gauduchon's plurigenera theorem all plurigenera of (M, J) vanish, cf. Proposition I.18 in [10] or [24]. Hence the Kodaira dimension of (M, J) is $-\infty$ [3].

If $b_1 = 0$, then $h^{0,1} = 0$ (see e.g. [3]). Hence, by the Castelnuovo criterion (cf. [3]) (M, J) must be a rational surface. The assertion about c_1^2 follows by the same arguments as those in the proof of Theorem 1.3 in [2].

If $b_1 = 1$, then Theorem 1.1 shows that (M, J) is a generalized Hopf manifold with respect to the Gauduchon metric. The recent classification of generalized Hopf surfaces [4] implies that the surface is a Hopf surface of class 1. **Q.E.D.**

For the rest of this section we assume that (M, g, J) is a Hermitian manifold, such that the canonical Weyl connection ∇^W is Hermitian-Weyl.

The Chern connection ∇^C of (M, g, J) is given by

$$g(\nabla_X^C Y, Z) = g(\nabla_X^W Y, Z) + \frac{1}{2}d\Omega(JX, Y, Z).$$

This, together with (2.2) and (3.10), yields

$$\nabla^C = \nabla^W + \frac{1}{2}\theta \otimes Id + \frac{1}{2}J\theta \otimes J,$$

where Id denotes the identity of TM . Consequently, the curvature tensors R^C and R^W of ∇^C and ∇^W are related by

$$(3.11) \quad R^C = R^W + \frac{1}{2}d(J\theta) \otimes J + \frac{1}{2}d\theta \otimes Id.$$

Notice that for $m > 2$ the last term in (3.11) is absent, since in this case $d\theta = 0$. Let

$$R^C(\Omega)(X, Y) = \frac{1}{2} \sum_{j=1}^{2m} g(R^C(e_j, J e_j)X, Y),$$

where $\{e_1, \dots, e_{2m}\}$ is an orthonormal frame of tangent vectors. We denote by k^C the symmetric, J -invariant bilinear form associated to $R^C(\Omega)$ and defined by

$$k^C(X, Y) = R^C(\Omega)(JX, Y).$$

Using (3.11) and the fact that $R^W \circ J = J \circ R^W$, we obtain

$$(3.12) \quad k^C = Ric^W + \frac{1}{2} \langle d(J\theta), \Omega \rangle g.$$

This equality is proved in [13] (Formula (14)) in dimension 4. The proof in higher dimensions is the same, using that $d\theta = 0$.

Proof of Theorem 1.3: We shall use Formula (2.8) in [26], according to which on every generalized Hopf manifold

$$(3.13) \quad d(J\theta) = |\theta|^2 \Omega + \theta \wedge J\theta.$$

Equalities (3.12) and (3.13) yield

$$(3.14) \quad k^C = Ric^W + (m-1)|\theta|^2 g.$$

Since $Ric^W \geq 0$ and $\theta \neq 0$, (3.14) implies $k^C > 0$. This allows us to apply the vanishing theorem for holomorphic $(p, 0)$ -forms [16, 9] and deduce that $h^{p,0} = 0$, $p = 1, 2, \dots, m$. On the other hand, the Hodge numbers on a $2m$ -dimensional generalized Hopf manifold satisfy the following relations [22]:

$$(3.15) \quad h^{m,0} = h^{m-1,0}, \quad h^{0,p} = h^{p,0} + h^{p-1,0}, \quad p \leq m-1, \quad 2h^{1,0} = b_1 - 1, \quad 2h^{0,1} = b_1 + 1.$$

From $h^{1,0} = 0$ and (3.15) we get $b_1 = 1$, $h^{0,1} = 1$. Further, (3.15) implies $h^{0,p} = 0$ for $p = 2, \dots, m-1$. Finally, the Kodaira-Serre duality and $h^{m,0} = 0$ lead to the vanishing of $h^{0,m}$, which completes the proof. **Q.E.D.**

Notice that if $2m = 4$ the result of Theorem 1.3 follows immediately from Corollary 1.2.

Acknowledgments The second author thanks to The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, where the final part of this work was done. Both authors thank V. Apostolov for his valuable comments and suggestions.

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